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J. Phys. A: Math. Gen. 24 (1991) 131–146. Printed in the UK

A class of exactly solvable potentials related to the Jacobi polynomials

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Received 31 July 1990

Abstract. We investigate a family of solvable potentials related to the Jacobi polynomials. This one-dimensional potential family depends on three parameters and is restricted to the domain $x \geq 0$, so it can be interpreted as the radial part of a central potential in three dimensions (with $l = 0$). Closed expressions are obtained for the bound state energy spectrum and the wavefunctions. The supersymmetric partner of this potential is also determined and it is found not to belong to the same potential family. It is shown that this potential family is a special subclass of the general six-parameter Natanzon potential class and similarities with another subclass, the Ginocchio potentials, are pointed out. Some aspects of supersymmetric quantum mechanics and shape invariance are also discussed in connection with the potential family under study.

1. Introduction

Solvable problems of non-relativistic quantum mechanics have always attracted much attention. The introduction of supersymmetric quantum mechanics (SUSYQM) (Witten 1981) had a strong impact on these studies and helped to view this field from a new angle. The main results of SUSYQM are simply stated (see, for example Cooper and Freedman 1983, Andrianov *et al* 1984, Sukumar 1985). If we know the ground state wavefunction of some potential $V_-(x)$, we can easily construct another potential $V_+(x)$, which is called the supersymmetric partner of $V_-(x)$, and which has the same energy eigenvalues, except for the ground state (which is missing from the spectrum of $V_+(x)$). In fact, a whole sequence of potentials can be generated in this way.

A further remarkable development was the introduction of the concept of shape invariance (Gendenshtein 1983). Many of the potentials related by supersymmetry were found to have similar shapes (i.e. to depend on the coordinate in a similar way), only the parameters appearing in them were different. It turned out that the energy spectrum and the wavefunctions can be determined by elementary calculations in this case. These shape-invariant potentials were identified with many of the already known potential problems of quantum mechanics with exact analytic solutions. Although originally it was thought (Gendenshtein 1983) that all solvable potentials were shape invariant, it turned out that shape invariance is not a general feature of solvable potentials (Cooper *et al* 1987). Later it was shown that shape invariance and SUSYQM are closely related to the factorization method (Schrödinger 1940a, b, 1941, Infeld and Hull 1951) (and therefore to the Darboux (1882) method of solving second-order

0305-4470/91/010131+16$03.50 © 1991 IOP Publishing Ltd

differential equations), in fact the domain of applicability of shape invariance turned out to be strictly identical (Montemayor and Salem 1989) with that of the factorization method of Infeld and Hull (1951).

Besides the results described above the combination of SUSYQM with 'traditional' approaches to solvable potentials proved to be fruitful. Using the concepts of SUSYQM the Natanzon (1971, 1979) potential class (which is a rather general class of solvable potentials related to the hypergeometric functions) was generalized to even wider classes of solvable potentials, in several ways (Cooper *et al* 1987, 1989). The question of solvability and shape invariance was studied, and although it was shown that the Natanzon potentials do not fulfil the shape-invariance requirement in general, the possibility of finding further shape-invariant potentials besides the already known ones cannot be excluded (Cooper *et al* 1987). In a previous study (Lévai 1989) we have also used an already known method (Bhattacharjie and Sudarshan 1962) to search for further shape-invariant potentials by linking it with the formalism of SUSYQM. As a by-product of these investigations we identified a class of solvable, but non-shape-invariant potentials. Here we discuss it in detail and show that it belongs to the Natanzon class. Although this potential class is not shape invariant, it preserved some features of shape-invariant potentials, and, since in terms of complexity it is between the general Natanzon potentials and their shape-invariant subclass (similarly to the Ginocchio (1984, 1985) potentials), it can help our understanding of the relationship between shape invariance and solvability.

The arrangement of this paper is as follows. In section 2 we review the main results of our previous study (Lévai 1989) necessary for the introduction of the new potential class and give a brief review of the Natanzon potentials. In section 3 we present a detailed study of this potential and its relation with the Natanzon and Ginocchio potentials. Finally, we summarize the results in section 4. In the appendix we give a compilation of the basic equations of SUSYQM and shape invariance necessary in sections 2 and 3.

2. Some aspects of solvable potentials

In a previous publication (Lévai 1989) we have performed a search for shape-invariant potentials using a simple method of finding solvable potentials and linking it with the formalism of SUSYQM. This method proved to be successful in identifying and classifying shape-invariant potentials. Since in that study we concentrated on shape-invariant potentials, the possibility of investigating non-shape-invariant, but solvable, potentials with the aid of this method was left open. Here we try to fill this gap (at least partly) by investigating a family of potentials related to the Jacobi polynomials. First we briefly review this simple method used previously, and its relation to the formalism of SUSYQM.

Consider the Schrödinger equation in one dimension (setting $\hbar = 2m = 1$):

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + (E - V(x))\Psi(x) = 0. \tag{2.1}$$

Its solutions are generally written as

$$\Psi(x) = f(x)F(g(x)) \tag{2.2}$$

where $F(g)$ is a special function satisfying a second-order differential equation

$$\frac{\mathrm{d}^2F}{\mathrm{d}g^2} + Q(g)\frac{\mathrm{d}F}{\mathrm{d}g} + R(g)F(g) = 0. \tag{2.3}$$

The form of $Q(g)$ and $R(g)$ is well defined for any special function $F(g)$. Substituting (2.2) into (2.1) and comparing it with equation (2.3) we get the following results:

$$\frac{g''}{(g')^2} + \frac{2f'}{g'f} = Q(g(x)) \tag{2.4}$$

and

$$\frac{f''}{f(g')^2} + \frac{E - V(x)}{(g')^2} = R(g(x)). \tag{2.5}$$

Now we can express $E - V(x)$ in terms of the other quantities in several ways:

$$E - V(x) = R(g(x))(g')^2 - f''/f \tag{2.6}$$

$$= R(g(x))(g')^2 - (f'/f)^2 - (f'/f)' \tag{2.7}$$

$$= \frac{g'''}{2g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + (g')^2\left(R(g(x)) - \frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}g} - \frac{1}{4}Q^2(g(x))\right). \tag{2.8}$$

The last equation was obtained after eliminating $f(x)$ and its derivatives using equation (2.4). We can use equation (2.8) to generate potentials by choosing $Q(g)$ and $R(g)$ (e.g. the type of the special function $F(g)$) and experimenting with various internal functions $g(x)$. (In principle we could try to proceed the other way round by substituting some $V(x)$ potential function in (2.8) and considering it a differential equation for $g(x)$, but there is not too much hope for solving it due to its complex nature.)

Equation (2.7) offers a convenient way to link this simple method with the equations of SUSYQM (see equation (A.2)), since whenever $R(g(x)) = 0$ holds, we get

$$E - V(x) = -W^2(x) + W'(x) \tag{2.9}$$

where

$$W(x) = -(\ln(f(x)))' \tag{2.10}$$

$$= -\frac{1}{2}Q(g(x))g'(x) + \frac{1}{2}\frac{g''(x)}{g'(x)}. \tag{2.11}$$

We derived the latter equation from the former one using

$$f(x) \simeq (g')^{-1/2}\exp\left(\frac{1}{2}\int^{g(x)} Q(g)\,\mathrm{d}g\right) \tag{2.12}$$

obtained from (2.4). Since (2.9) is the standard expression for potentials in SUSYQM, inspecting the structure of $R(g)$ can help us to decide which special functions $F(g)$ are more appropriate in SUSYQM applications. In our previous study (Lévai 1989) we used the orthogonal polynomials as special functions, since the structure of $R(g)$ in

that case is such that it vanishes for $n = 0$. (As a counterexample we mentioned the Bessel functions, for which $R(g) = 1-\nu^2/g^2$ holds, so its application in SUSYQM seems less promising, since $R(g)$ cannot be set to zero, irrespective of the parameter ν.)

In the case of the $P_n^{(\alpha,\beta)}(g)$ Jacobi polynomials we have

$$Q(g) = \frac{\beta-\alpha}{1-g^2} - (\alpha+\beta+2)\frac{g}{1-g^2} \tag{2.13}$$

and

$$R(g) = \frac{n(n+\alpha+\beta+1)}{1-g^2} \tag{2.14}$$

so equation (2.8) is written as

$$\begin{aligned} E - V(x) = {} & \frac{1}{2}\frac{g'''}{g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + \frac{(g')^2}{1-g^2}\left(n+\frac{\alpha+\beta}{2}\right)\left(n+\frac{\alpha+\beta+1}{2}\right) \\ & + \frac{(g')^2}{(1-g^2)^2}\left(\left(1-\frac{\alpha+\beta}{2}\right)\left(1+\frac{\alpha+\beta}{2}\right) - \left(\frac{\beta-\alpha}{2}\right)^2\right) \\ & + \frac{(g')^2 g}{(1-g^2)^2}\frac{1}{2}(\beta-\alpha)(\beta+\alpha). \end{aligned} \tag{2.15}$$

Now, observing that there is a constant term (E) on the left-hand side of (2.15) we require some of the terms on the right-hand side to be constant. This requirement amounts to setting up differential equations for $g(x)$. If we require one of those terms which contain α, β and n to be constant, this differential equation is a first-order one and it can be solved in a straightforward way. In our previous study (Lévai 1989) we identified two potential classes related to the Jacobi polynomials setting the first and the second such terms of (2.15) to constants. These potential classes were labelled by the symbols PI and PII, and they turned out to be the well known shape-invariant potentials related to the hypergeometric functions. (We identified, however, one potential belonging to the PII class, which was missing from other works discussing shape-invariant potentials.) Similar treatment of the $L_n^{(\alpha)}(g)$ generalized Laguerre and the $H_n(g)$ Hermite polynomials resulted in the identification of every single known shape-invariant potentials. Although this simple method of generating internal functions $g(x)$ does not guarantee that we find every single potential related to the Jacobi polynomials, it is a convenient way of finding the most obvious ones. Later we shall see how it can be generalized in order to get a wider class of solvable potentials.

On the basis of the procedure described above we introduced a straightforward classification scheme of shape-invariant potentials, which turned out to be basically the same as that of Infeld and Hull (1951) based on the factorization method, and that of Miller (1968) originating from the Lie theory of special functions, but it surprisingly differed from the classification of Cooper *et al* (1987) which was designed following the principles of SUSYQM.

The remaining terms of equation (2.15) were not studied in detail. We have neglected the differential equation

$$\frac{(g')^2 g}{(1-g^2)^2} = C \tag{2.16}$$

referring to the fact that solving it we can get only the inverse $x(g)$ function in closed form. As we shall see later this is not a serious problem from the point of view of solvability. In the next section we shall study the potential class obtained from equation (2.16). In analogy with the PI and PII shape-invariant potential classes related to the Jacobi polynomials, we call this potential class the PIII potential family. A further possibility is to take combinations of terms appearing on the left-hand side of equation (2.15) and setting up more complex differential equations for $g(x)$ (with several parameters for example), but we do not discuss this possibility further here.

The simple method of obtaining solvable potentials reviewed in this section has been known for a long time (except, of course, its relationship with SUSYQM). Bhattacharjie and Sudarshan (1962) applied it to the hypergeometric, confluent hypergeometric and the Bessel functions. The combination of SUSYQM and some other traditional approaches to solvable potentials has also proved to be fruitful. Using the techniques of SUSYQM the Natanzon potentials (Natanzon 1971, 1979) which form a rather general potential class related to the hypergeometric functions, were generalized to an even wider class of solvable potentials (Cooper *et al* 1987).

The general form of the Natanzon potentials contains six parameters:

$$\begin{aligned} V(x) = {} & [fz(x)(z(x)-1) + h_0(1-z(x)) + h_1 z(x) + 1]/R(z(x)) \\ & + \{a + [a + (c_1 - c_0)(2z(x)-1)]/(z(x)(z(x)-1)) \\ & - 5\Delta/4R(z(x))\} z^2(x)(1-z(x))^2/R^2(z(x)) \end{aligned} \tag{2.17}$$

where

$$R(z(x)) = az^2(x) + (c_1 - c_0 - a)z(x) + c_0 \tag{2.18}$$

and

$$\Delta = (a - c_0 - c_1)^2 - 4c_0c_1. \tag{2.19}$$

The function $z(x)$ must be such that it maps the full x axis to the $z = [0,1]$ interval and it obeys the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2z(x)(1-z(x))}{R^{1/2}(z(x))}. \tag{2.20}$$

The well known shape-invariant potentials correspond to some simple choice of $R(z)$. The energy eigenvalues ϵ_n are obtained from the following equation:

$$\begin{aligned} 2n+1 &= (1 - a\epsilon_n + f)^{1/2} - (1 - c_0\epsilon_n + h_0)^{1/2} - (1 - c_1\epsilon_n + h_1)^{1/2} \\ &\equiv \alpha_n - \beta_n - \delta_n \end{aligned} \tag{2.21}$$

while the wavefunctions are written as

$$\Psi_n \simeq R^{1/4}(z(x)) z^{\beta_n/2}(x)(1-z(x))^{\delta_n/2}\, {}_2F_1(-n, \alpha_n - n; 1 + \beta_n; z(x)). \tag{2.22}$$

We shall use these equations to discuss the relationship of the PIII and the Natanzon potentials.

3. Detailed study of the PIII potential family

As discussed in the previous section, solving the differential equation (2.16) we cannot express the function $g(x)$ in closed form, only the inverse $x(g)$ function can be determined:

$$x(g) = C^{-1/2}\left(\tanh^{-1}(g^{1/2}) - \tan^{-1}(g^{1/2})\right). \tag{3.1}$$

It is easy to show that this function is real for $g \geq 0$, but it is purely imaginary for $g < 0$, since

$$x(-g) = -i\, x(g) \tag{3.2}$$

so negative values of g must be excluded if we want to interpret x as the coordinate. $x(g)$ maps the $g = [0,1]$ interval on the half axis $x = [0,\infty)$ in a strictly monotonic way. Its long-range behaviour ($x \gg 1$) is similar to that of the $\tanh^{-1}(g^{1/2})$ function, while for small x (and g) its behaviour is determined by the first term of the series expansion

$$x(g) = C^{-1/2} \sum_{k=0}^{\infty} \frac{g^{2k+3/2}}{2k+3/2} \quad . \tag{3.3}$$

so it can be approximated with $x \simeq C^{-1/2} 2g^{3/2}/3$. On the basis of these arguments we can see that the approximate behaviour of $g(x)$ is $g(x) \simeq C^{1/3}(3x/2)^{2/3}$ near the origin, while its long-range behaviour is approximately $g(x) \simeq \tanh^2(C^{1/2}x)$. Since the $g(x)$ function can be computed numerically to any desired accuracy, we can also determine the wavefunctions and the potential (which depend on $g(x)$) for any value of x. Therefore it is enough to calculate these quantities as a function of g, and later we can inspect their approximate dependence on x whenever necessary.

Now let us turn our attention to the explicit determination of the potential in terms of g, using the procedure described in the preceding section. In order to do this, first we introduce the new parameters $p_n = (\alpha_n + \beta_n + 1)/2$ and $q_n = (\beta_n - \alpha_n)/2$. (Here we assume that these variables can explicitly depend on n.) Equation (2.15) can now be rewritten as

$$\begin{aligned} E_n - V(g(x), p_n, q_n) = C\big[&\tfrac{5}{16} g^{-3}(x) + g^{-1}(x)\left(-q_n^2 + p_n + n(n+2p_n) + \tfrac{5}{8}\right) \\ &+ g(x)\left(\tfrac{1}{16} - (p_n+n)^2\right) + q_n(2p_n - 1)\big]. \end{aligned} \tag{3.4}$$

In order to have the same potential function for any n we now introduce explicit expressions for p_n and q_n, which set the coefficients of the g-dependent terms in (3.4) to n-independent values. To this end we first take equation (3.4) for $n = 0$, determine the expression $[E_n - V(g(x), p_n, q_n)] - [E_0 - V(g(x), p_0, q_0)]$ and set the coefficients of the g-dependent (coordinate dependent) terms to zero. Doing so, we get the following equations for p_n and q_n:

$$\begin{aligned} &(p_n + n)^2 = p_0^2 \\ &-q_n^2 + p_n + n(n+2p_n) = -q_0^2 + p_0. \end{aligned} \tag{3.5}$$

Introducing the notation $p \equiv p_0$ and $q \equiv q_0$, we can readily determine the explicit expression for the parameters p_n and q_n:

$$p_n = p - n$$
$$q_n = \begin{cases} q & \text{for } n = 0 \\ \left(q^2 + (p - \frac{1}{2})^2 - (p - \frac{1}{2} - n)^2\right)^{1/2} & \text{for } n > 0. \end{cases} \tag{3.6}$$

The final form of the potential function, which is now free from the quantum number n, is the following:

$$V(x) = C\left[-q(2p-1) + g(x)\left(p^2 - \tfrac{1}{16}\right) + g^{-1}(x)\left(q^2 - p + \tfrac{5}{8}\right) - \tfrac{5}{16}g^{-3}(x)\right] \tag{3.7}$$

while for the energy expression (which originates from the constant term in equation (3.4)) we get

$$E_n = 2C\left(p - \tfrac{1}{2} - n\right)\left[q^2 + \left(p - \tfrac{1}{2}\right)^2 - \left(p - \tfrac{1}{2} - n\right)^2\right]^{1/2} - 2C\left(p - \tfrac{1}{2}\right)q. \tag{3.8}$$

(This expression is valid only for $n > 0$, while for $n = 0$ we, of course, get $E_0 = 0$.) Due to the square root in equation (3.8) the stucture of the energy spectrum is more complex than for other solvable potentials. From equations (3.7) and (3.8) one can see that the PIII potential family depends on three parameters, two of which (p and q) determine the shape of the potential (as we shall see later) and one (C) is a scaling factor of the energy (and the coordinate).

We can use equation (2.11) to express the $W(x) = W(g(x))$ superpotential for this class of potentials:

$$W(g(x)) = C^{1/2}\left[\left(p - \tfrac{1}{4}\right)g^{1/2}(x) - qg^{-1/2}(x) - \tfrac{1}{4}g^{-3/2}(x)\right]. \tag{3.9}$$

The supersymmetric partner of $V(x) \equiv V_-(x)$, denoted by $V_+(x)$ can also be calculated now:

$$V_+(x) = C\left[\tfrac{7}{16}g^{-3}(x) + qg^{-2}(x) + g^{-1}(x)\left(q^2 - \tfrac{3}{8}\right) + g(x)\left(p - \tfrac{1}{4}\right)\left(p - \tfrac{3}{4}\right) - 2pq\right]. \tag{3.10}$$

It is easy to prove that $V_+(x)$ and $V(x)$ do not fulfil the shape invariance condition (A.8), so this new potential family is an example for non-shape-invariant solvable potentials. (An indication for this is, for example, the different numerical coefficient of the terms containing $g^{-3}(x)$ in equations (3.7) and (3.10), and also the presence of the $g^{-2}(x)$ term in equation (3.10), without equivalent in equation (3.7).)

Let us first inspect the behaviour of the potential $V(x) = V(g(x))$. It is easy to prove that both $V(x)$ and $V_+(x)$ go to the same value in the $x \to \infty$ (i.e. $g \to 1$) limit. This common value depends only on $p - q$:

$$V(x = \infty) = V_+(x = \infty) = C\left(p - q - \tfrac{1}{2}\right)^2 = C\alpha^2. \tag{3.11}$$

It is also easy to determine the approximate behaviour of these potentials near the origin. In particular we get the following approximations (from (3.3)) in the $x \to 0$ ($g \to 0$) limit:

$$V(x) \simeq -\tfrac{5}{36}x^{-2} \tag{3.12}$$

for $V(x)$ and

$$V_+(x) \simeq \tfrac{7}{36}x^{-2} \tag{3.13}$$

for $V_+(x)$. Note, that these equations are independent of the potential parameters.

The considerations presented above suggest that these potentials should be viewed as the radial part of a spherical potential (with $l = 0$). This development naturally raises the question of whether the particle 'falls' into the attractive $V(x)$ potential well at the origin or not. The answer is definitely 'not', since we know (Landau and Lifshitz 1977), that the falling of a particle into the potential $V(x) \simeq -\gamma/x^2$ can occur only if $\gamma > \frac{1}{4}$ holds, while in this case we have $\gamma = \frac{5}{36} < \frac{1}{4}$.

Besides the $x \to 0$ ($g \to 0$) and $x \to \infty$ ($g \to 1$) cases we can study the behaviour of the $V(x)$ potential in the domain between these limits as the function of the parameters p and q. In order to determine whether it has any minima or maxima, we calculate the derivative of $V(x)$ with respect to x:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = C^{1/2}(1-g^2)g^{-1/2}\frac{\mathrm{d}V}{\mathrm{d}g}. \tag{3.14}$$

As we expect, the derivative goes to $+\infty$ for $g \to 0$ and to zero for $g \to 1$. The behaviour of the potential in between these two limits can be determined inspecting the $\mathrm{d}V/\mathrm{d}g$ derivative:

$$\frac{\mathrm{d}V}{\mathrm{d}g} = C\left[\left(p^2 - \tfrac{1}{16}\right) + g^{-2}\left(-q^2 + p - \tfrac{5}{8}\right) + \tfrac{15}{16}g^{-4}\right] = 0. \tag{3.15}$$

This is a second-order algebraic equation for g^{-2} and its solutions are

$$(g_{1,2})^{-2} = \tfrac{8}{15}\left[q^2 - p + \tfrac{5}{8} \pm \left(\left(q^2 - p + \tfrac{5}{8}\right)^2 - \tfrac{15}{4}\left(p^2 - \tfrac{1}{16}\right)\right)^{1/2}\right]. \tag{3.16}$$

Equation (3.16) can help us to find the maxima and minima of the $V(x)$ potential for given parameters p and q. Since the domain of definition of $V(g)$ is the interval $g = [0,1]$, not every solution of equation (3.16) gives the location of a minimum or a maximum of $V(x)$. Inspecting the real roots of (3.16) in the $g^{-2} = [1,\infty)$ domain we can establish the following. If the discriminant of (3.16) is negative, $V(x)$ is strictly monotonics, since it has no extrema. If the discriminant is zero (i.e. if we have two identical roots of (3.16)) and $(g_1)^{-2} = (g_2)^{-2} \leq 1$, then there are still no extrema, since in this case $g_1 = g_2 \geq 1$, so the roots are outside the domain of definition of $V(g)$. If $g_1 = g_2 < 1$, $V(g)$ (and therefore $V(x)$) has a point of inflexion. If the discriminant is positive, and both g_1 and g_2 are located in the domain $g = (0,1)$, $V(x)$ first takes a maximum then a minimum and increasingly tends to $V(x = \infty)$ with increasing x. If only the smaller root (g_2) lies within the interval $g = (0,1)$, $V(x)$ has only a maximum and decreasingly tends to $V(x = \infty)$, while if both roots lie outside the specified domain, $V(x)$ has no local extrema. In figure 1 examples are presented for each pattern described in this paragraph. Here we plotted V as the function of g (and not x), but this does not affect the general pattern of local minima and maxima. In figure 2 we present a 'map' of the (p,q) plane divided into domains corresponding to potential types described above. It can be seen from figure 2 that potentials with no

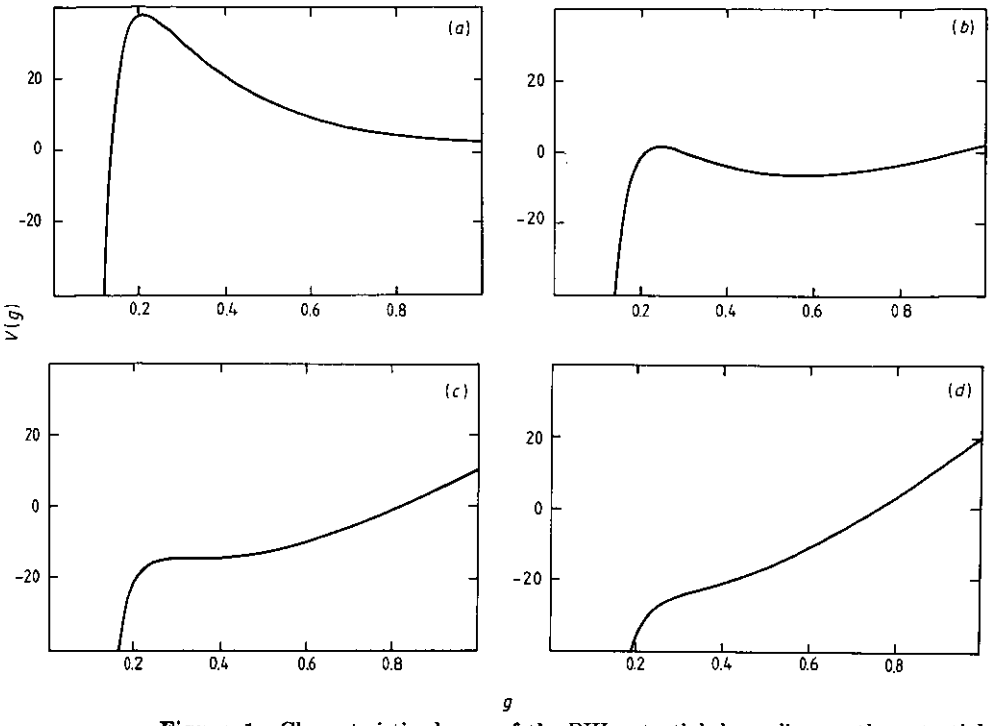

Figure 1. Characteristic shapes of the PIII potential depending on the potential parameters p and q; (*a*) $p = 4$, $q = 5$, (*b*) $p = 7$. $q = 5$, (*c*) $p = 8.279$, $q = 5$, (*d*) $p = 10$, $q = 5$. The potential V is plotted against the variable g (rather than x), but due to the strictly monotonous nature of $g(x)$ this does not affect the general trend (maxima and minima) of the potential. Independently of the parameters the potentials go to $-\infty$ as $V(g) \simeq -\frac{5}{16}g^{-3}$ in the $g \to 0$ limit.

extrema are located near the $q = 0$ line, while potentials with only a maximum can be found near the $p = -1/2$ line.

Similar treatment of the $V_+(x)$ supersymmetric partner potential is also possible, but in this case we have to analyse the roots of a third-order algebraic equation to find the extrema of the potential, so we shall not study this case in detail.

Now let us turn our attention to the wavefunctions. Using equation (2.12) they can be written (without normalization) as

$$\begin{aligned}\Psi_n(g(x)) &\simeq g^{1/4}(x)(1+g(x))^{\beta_n/2}(1-g(x))^{\alpha_n/2}P_n^{(\alpha_n,\beta_n)}(g(x)) \\ &= g^{1/4}(x)(1+g(x))^{(p_n+q_n-1/2)/2}(1-g(x))^{(p_n-q_n-1/2)/2} \\ &\quad \times P_n^{(p_n-q_n-1/2,p_n+q_n-1/2)}(g(x)). \qquad (3.17)\end{aligned}$$

As we can expect on the basis of arguments presented earlier (Landau and Lifshitz 1977), this function can be approximated with $x^{1/6}$ near the origin. In the $g = [0,1]$ interval each component of $\Psi_n(g)$ is bounded from above, with the exception of the function $(1-g)^{\alpha_n/2}$, which tends to ∞ for $g \to 1$, if α_n is negative. We can use this fact to inspect when we can get regular, normalizable solutions, i.e. to determine the number of bound states for arbitrary values of p and q. It is easy to prove that whenever $\alpha_n = p_n - q_n - \frac{1}{2} < 0$ holds for a given n, the same will be valid for α_{n+1} as well, so the bound states are characterized by the quantum numbers $n < n_{\max}$.

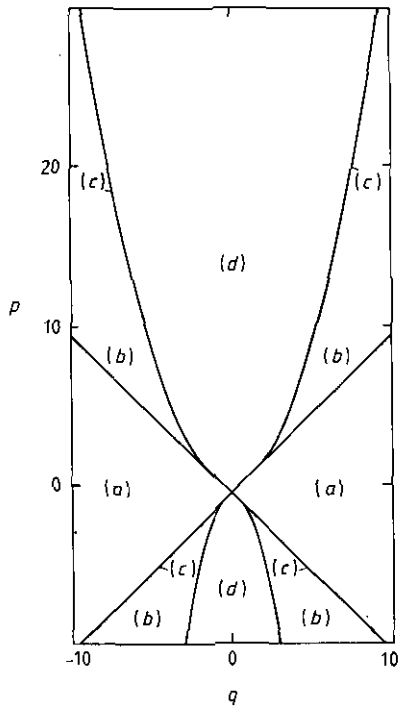

Figure 2. The domains of the (p,q) plane corresponding to the characteristic shapes of the potential displayed in figure 1.

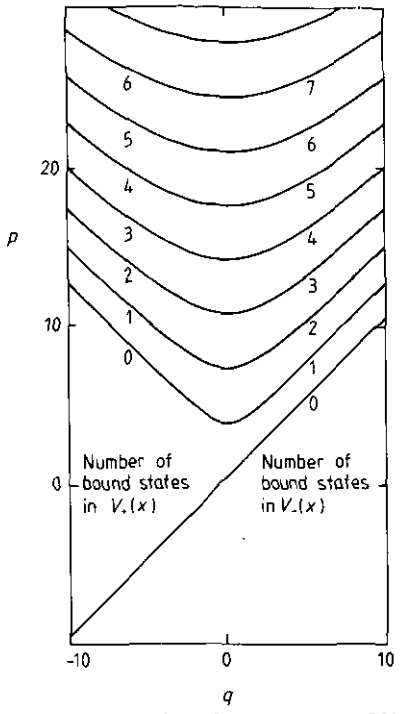

Figure 3. The number of bound states of $V_-(x)$ (right) and $V_+(x)$ (left) depending on the values of the parameters p and q.

Another development which can be seen from equation (3.17) is that wavefunctions with $\alpha_n = 0$ tend to a finite constant value for $x \to \infty$, so they are not normalizable. In order to normalize the wavefunction we have to compute the following integral:

$$\int_0^\infty |\Psi_n(x)|^2 \, \mathrm{d}x = C^{-1} \int_0^1 g(1+g)^{\beta_n - 1}(1-g)^{\alpha_n - 1} \left(P_n^{(\alpha_n,\beta_n)}(g) \right)^2 \mathrm{d}g. \tag{3.18}$$

This integral can be computed numerically. It can be seen from the behaviour of the integrand near $g = 1$ that this integral is finite if $\alpha_n > 0$ holds.

The results obtained in the previous paragraph can be summarized in figure 3, which is another 'map' of the (p,q) plane showing the number of bound states in $V(x)$ (and in $V_+(x)$) assigned to specified values of p and q. From figure 3 it can be seen that a large area corresponds to potentials with no bound states: whenever $\alpha = p - q - \frac{1}{2} \leq 0$ holds there will be no bound states of $V(x)$. It is interesting to compare figures 2 and 3. One can see, for example, that potentials having only a maximum (case (a) in figures 1 and 2) can support only one bound state for $q < 0$ and none for $q > 0$, while potentials having both a maximum and a minimum (case (b)) and potentials without extrema (case (d)) can support any specified number of bound states. The lines separating the sections of figure 3 originate from the following equation (obtained from $\alpha_n = 0$):

$$p = \begin{cases} \frac{1}{2} + 2n + (2n^2 + q^2)^{1/2} & n > 0 \\ \frac{1}{2} + q & n = 0. \end{cases} \tag{3.19}$$

The upper limit of n for regular solutions is the following:

$$p - \frac{1}{2} - \frac{1}{\sqrt{2}}\left[q^2 + \left(p - \frac{1}{2}\right)^2\right]^{1/2} > n. \tag{3.20}$$

(This equation is also obtained from $\alpha_n > 0$.)

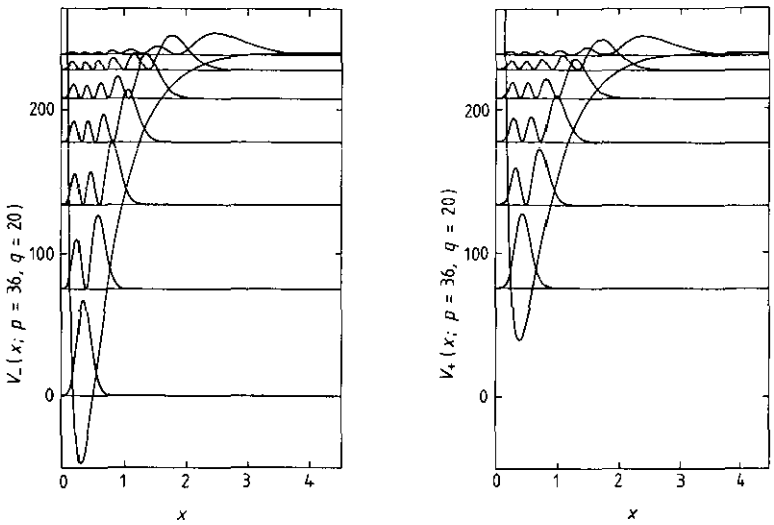

Figure 4. The PIII potential and its supersymmetric partner for the parameters $p = 36$, $q = 20$ and $C = 1$, displayed together with the squared modulus of the wavefunctions. (The true behaviour of $V_-(x)$ cannot be seen near the origin. $V_-(x)$ has a maximum at $x = 0.0077$ and tends to $-\infty$ in the $x \to 0$ limit.)

One can see from the structure of the energy spectrum (3.8) that its highest possible value is the same as $V(x = \infty)$, which means that carefully tuning p and q we can obtain an energy level arbitrarily close to $V(x = \infty)$. This can be seen for example from the following relation:

$$V(x = \infty) - E_n = C\left(p_n - \tfrac{1}{2} - q_n\right)^2 = C\alpha_n^2 \tag{3.21}$$

which means that approaching zero with α_n we can set this difference as small as we wish. At the same time, due to the presence of the term $(1 - g(x))^{\alpha_n/2}$, the wavefunction slowly 'flows out' from the potential well.

We can also determine the explicit form of the wavefunctions $\Psi_n^{(+)}(x)$ associated with the $V_+(x)$ supersymmetric partner potential. In terms of supersymmetric quantum mechanics (see equations (A.4) and (A.7)) they can be written as

$$\begin{aligned} &\Psi_n^{(+)}(x) \simeq A\ \Psi_{n+1}^{(-)}(x) \\ \equiv\ &A\ \Psi_{n+1}(x) \\ \equiv\ &\left(\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right)\Psi_{n+1}(x) \end{aligned}$$

$$= C^{1/2}g^{-1/4}(x)(1+g(x))^{(\beta_{n+1})/2}(1-g(x))^{(\alpha_{n+1})/2}$$
$$\times\left[\left(q_{n+1}-q-(n+1)\frac{2q_{n+1}}{2p-1}\right)P_{n+1}^{(\alpha_{n+1},\beta_{n+1})}(g(x))\right.$$
$$\left.+\,2\frac{(p_{n+1}-1/2)^2-q^2}{2p-1}P_n^{(\alpha_{n+1},\beta_{n+1})}(g(x))\right]. \tag{3.22}$$

Here we used the recurrence relation for the derivative of the Jacobi polynomials (Abramowitz and Stegun 1970), and as a result of this, the wavefunction is the linear combination of two Jacobi polynomials. (Note, that the parameters of the two Jacobi polynomials are the same, α_{n+1} and β_{n+1}, but their indices differ.) The structure of these wavefunctions shows that $V_+(x)$ does not belong to the PIII potential family. Studying the regularity of the wavefunctions in (3.22) we find that the number of bound states supported by $V_+(x)$ is $m-1$, if $V(x)$ has m bound states. (This is what we also expect from the theory of SUSYQM: see equation (A.6).) The 'map' of the (p,q) plane, displayed in figure 3 can be used for $V_+(x)$ as well: on its left-hand side we presented the number of bound states supported by the potential $V_+(x)$ for any values of p and q. It is clear from equation (3.19) that $(2p-1)^{-1}$ does not affect the regularity of the wavefunctions, since bound states of $V_+(x)$ can occur only for $p \geq 5/2+\sqrt{2}$.

As an illustrative example, in figure 4 we present $V(x)$ and its supersymmetric partner, $V_+(x)$, for the $p=36$, $q=20$ case. These potentials support 7 and 6 bound states, respectively, and belong to domain (*b*) in figure 2 We also present the squared modulus of the wavefunction belonging to these states. The true behaviour of $V(x)$ near the origin cannot be seen in figure 4: in fact it has a local maximum at $x=0.0077$ and tends to $-\infty$ for $x \to 0$. (Potentials supporting a large number of states belong to those domains of the (p,q) plane where the location of the extrema is close to the origin.) Using the standard techniques of supersymmetric quantum mechanics we can generate further potentials from $V_+(x)$, since we know its ground state wavefunction. The explicit expression of the wavefunctions of these potentials contain even more Jacobi polynomials. Here we shall not study them further.

Finally, we inspect the relation of the PIII potential family to the Natanzon class of potentials (Natanzon 1971, 1979), which depends on six parameters and which has hypergeometric functions in the solutions of the corresponding Schrödinger equation. It can be shown easily that similarly to the Ginocchio class of potentials (Ginocchio 1984, 1985) (depending on two parameters, λ and ν) the three-parameter PIII potential family is also a subclass of the general Natanzon class of potentials; however the restriction imposed on the six parameters of the Natanzon potentials differs in the two cases. In order to show this, first we use the well known relationship between the Jacobi polynomials and the hypergeometric functions (Abramowitz and Stegun 1970). Using (for example) the transformation formula

$$P_n^{(\alpha,\beta)}(g)=\binom{n+\alpha}{n}\,{}_2F_1\left(-n,n+\alpha+\beta+1;\alpha+1;\frac{1-g}{2}\right) \tag{3.23}$$

we get the following relations between the six parameters of the general Natanzon potentials and the three parameters (p, q and C) of the PIII potential family:

$$a=0 \qquad c_0=\frac{1}{C} \qquad c_1=-\frac{1}{C} \qquad f=4p^2-1$$
$$1+h_0-\frac{\epsilon_0}{C}=\left(p-\tfrac{1}{2}-q\right)^2 \qquad 1+h_1+\frac{\epsilon_0}{C}=\left(p-\tfrac{1}{2}+q\right)^2 \tag{3.24}$$

where ϵ_0 is the energy of the ground state in the corresponding Natanzon potential, and (contrary to the SUSYQM) it is not zero in general, rather it can be determined from equation (2.21) for the $n = 0$ case. Now it is

$$\frac{\epsilon_0}{C} = \tfrac{1}{2}(h_0 - h_1) + [(1+f)^{1/2} - 1]\{\tfrac{1}{2}(h_0 + h_1 + 2) - \tfrac{1}{4}[(1+f)^{1/2} - 1]^2\}^{1/2}. \qquad (3.25)$$

Similarly to the Ginocchio potential class (see Cooper *et al* (1987) for the details) we have one parameter (C) which is related to the subset of parameters (a, c_0 and c_1) appearing in the differential equation (2.20) and which acts as a scaling parameter, but we have two parameters (p and q) related to the other subset of parameters h_0, h_1 and f, determining the actual shape of the potential. Another similarity between the PIII and the Ginocchio potentials is that one of the three parameters c_0, c_1 and a is strictly zero, which enables us to express the energy from equation (2.21) in a relatively simple way. (This parameter is a in the example presented above, but using other transformations linking the Jacobi polynomials with the hypergeometric functions (see, for example, Abramowitz and Stegun 1970) we could also get other sets of relations instead of (3.24), among which we would have $c_0 = 0$ or $c_1 = 0$ as well.) In the case of shape-invariant potentials only one of the three parameters can differ from zero (see, for example, Cooper *et al* 1987), so the determination of the energy spectrum from (2.21) becomes straightforward for these potentials.

These arguments help us to illuminate the problem of finding shape-invariant potentials among the general Natanzon class ones. The shape-invariance requirement (A.8) imposes strong constraints on the structure of the energy spectrum and on the relation between the parameters appearing in the expression of the energy spectrum and the potentials (see (A.10), (A.11)). Therefore we expect that the complex spectra of the potentials like the PIII and Ginocchio class (arising from the fact that we have more than one non-zero parameters among c_0, c_1 and a) cannot reproduce this simple pattern. (This is also discussed by Cooper *et al* (1987).) Note, however, that in the PIII case formally we can express the energy spectrum in a form required by shape invariance (see (A.10)),

$$E_n = \sum_{k=1}^{n} R(p_k, q_k) \qquad (3.26)$$

where

$$R(p_k, q_k) = 2C\left(p_k - \tfrac{1}{2}\right) q_k - 2C\left(p_{k-1} - \tfrac{1}{2}\right) q_{k-1} \qquad (3.27)$$

but since the functional (in (A.11)) which links the parameter sets $\{p_k, q_k\}$ and $\{p_{k-1}, q_{k-1}\}$ is too complex (see equation (3.6)) we cannot get a shape-invariant potential. It is worth mentioning here that, contrary to the Ginocchio potential, which goes to a shape-invariant (Pöschl–Teller) potential in the $\lambda = 1$ limit, the PIII potential has no shape-invariant limiting case. This is in connection with the fact that due to the relatively complex way λ is related to c_0, c_1 and a (i.e. $c_0 = 0$, $c_1 = 1/\lambda^4$, $a = (1 - \lambda^2)/\lambda^4$, see Cooper *et al* (1987)), the $\lambda = 1$ choice plays a distinguished role among the possible values, while the parameter C has no similar special value (see (3.24)).

4. Summary and conclusions

We have investigated a family of solvable potentials related to the Jacobi polynomials as solutions of the corresponding Schrödinger equation. Although this potential family was found in connection with investigations concerning shape-invariant potentials (Lévai 1989), it turned out not to be shape invariant.

These potentials depend on three parameters, one of which is a scaling parameter of the energy (and the coordinate), while the two other determine the shape of the potentials. Characteristic shapes (minima and maxima) of the potentials depending on these two parameters have been identified. The coordinate is restricted to positive values only, so the PIII potentials can be interpreted as the radial part of central potentials in three dimensions. A characteristic feature of these potentials is their behaviour near the origin. Independently of the parameters they tend to $-\infty$ as $V(x) \simeq -\frac{5}{36}x^{-2}$. This numerical constant is not strong enough for the particle to 'fall' into the attractive potential (Landau and Lifshitz 1977). It was shown that these potentials can support only a finite number of bound states, depending on the actual value of the parameters. The energy spectrum and the corresponding wavefunctions have been determined (for $l = 0$).

The supersymmetric partner of the general PIII potential has also been studied. Since the wavefunctions of the supersymmetric partner potential can be written as linear combinations of two Jacobi polynomials, these potentials cannot belong to the PIII potential family.

It was shown that the PIII potentials form a special subclass of the six-parameter Natanzon potential class (Natanzon 1971, 1979). We have also pointed out similarities and differences between the PIII family and the Ginocchio class of potentials (Ginocchio 1984, 1985), which also forms a special (two-parameter) subclass of the Natanzon potentials. These considerations may help us to identify further subclasses of the Natanzon potentials and may serve as an aid for further investigations concerning the relationship between shape invariance and solvability.

Appendix

Here we briefly review the basic results of SUSYQM used in the main body of the paper. Introduction to the general theory of SUSYQM can be found, for example, in the works of Cooper and Freedman (1983), Andrianov *et al* (1984) and Sukumar (1985).

In SUSYQM two one-dimensional Hamiltonians related by supersymmetry can be written as ($\hbar = 2m = 1$)

$$H_{\pm} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{\pm}(x) \tag{A.1}$$

where $V_-(x)$ and $V_+(x)$ are expressed using the superpotential $W(x)$:

$$V_{\pm}(x) = W^2(x) \pm W'(x). \tag{A.2}$$

The partner Hamiltonians can be factorized as

$$H_- = A^\dagger A \qquad H_+ = AA^\dagger \tag{A.3}$$

where

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x) \qquad A^{\dagger} = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x). \tag{A.4}$$

In the case of unbroken supersymmetry the ground state of H_- has zero energy ($E_0^{(-)} = 0$) and the ground state wavefunction is related to the superpotential $W(x)$ by

$$W(x) = -(\ln \Psi_0^{(-)}(x))'. \tag{A.5}$$

The energy eigenvalues of H_- and H_+ are identical, except for the ground state:

$$E_n^{(+)} = E_{n+1}^{(-)} \qquad n = 0,\ 1, \ldots \qquad E_0^{(-)} = 0. \tag{A.6}$$

The eigenfunctions of H_- and H_+ (denoted by $\Psi_n^{(-)}(x)$ and $\Psi^{(+)}(x)$, respectively) are connected by the operators A and $A^{\dagger}$:

$$\begin{aligned} A\Psi_{n+1}^{(-)}(x) &= (E_n^{(+)})^{1/2}\Psi_n^{(+)}(x) \\ A^{\dagger}\Psi_n^{(+)}(x) &= (E_n^{(+)})^{1/2}\Psi_{n+1}^{(-)}(x). \end{aligned} \tag{A.7}$$

The concept of shape invariance (Gendenshtein 1983) relates supersymmetric partner potentials through the parameters appearing in them. Potentials are called shape invariant if their dependence on x, the coordinate is similar and they differ only in the potential parameters. In particular, $V_+(x,a)$ and $V_+(x,a)$ are shape invariant if they satisfy the shape-invariance relationship

$$V_+(x,a_0) - V_-(x,a_1) \equiv W^2(x,a_0) + W'(x,a_0) - W^2(x,a_1) + W'(x,a_1) = R(a_1) \tag{A.8}$$

where $R(a)$ depends only on the parameters a (and not on the coordinate x), and the parameters a_1 and a_0 are related by a functional:

$$a_1 = f(a_0). \tag{A.9}$$

A consequence of shape invariance is that the energy spectrum can be written in terms of the constants $R(a_k)$:

$$E_n^{(-)} = \sum_{k=1}^{n} R(a_k) \tag{A.10}$$

where a_k is obtained from a_0 by acting on it with the functional f k times:

$$a_k = f^k(a_0). \tag{A.11}$$

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